

# On some intermediate mean values

Slavko Simic

Mathematical Institute SANU, Kneza Mihaila 36, 11000 Belgrade, Serbia

E-mail: ssimic@turing.mi.sanu.ac.rs

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**Abstract** We give a necessary and sufficient mean condition for the quotient of two Jensen functionals and define a new class  $\Lambda_{f,g}(a, b)$  of mean values where  $f, g$  are continuously differentiable convex functions satisfying the relation  $f''(t) = tg''(t), t \in \mathbb{R}^+$ . Then we asked for a characterization of  $f, g$  such that the inequalities  $H(a, b) \leq \Lambda_{f,g}(a, b) \leq A(a, b)$  or  $L(a, b) \leq \Lambda_{f,g}(a, b) \leq I(a, b)$  hold for each positive  $a, b$ , where  $H, A, L, I$  are the harmonic, arithmetic, logarithmic and identric means, respectively. For a subclass of  $\Lambda$  with  $g''(t) = t^s, s \in \mathbb{R}$ , this problem is thoroughly solved.

## 1. Introduction

1. 1 It is said that the mean  $P$  is intermediate relating to the means  $M$  and  $N$ ,  $M \leq N$  if the relation

$$M(a, b) \leq P(a, b) \leq N(a, b),$$

holds for each two positive numbers  $a, b$ .

It is also well known that

$$\min\{a, b\} \leq H(a, b) \leq G(a, b) \leq L(a, b) \leq I(a, b) \leq A(a, b) \leq S(a, b) \leq \max\{a, b\}, \quad (1)$$

where

$$H = H(a, b) := 2(1/a + 1/b)^{-1}; \quad G = G(a, b) := \sqrt{ab}; \quad L = L(a, b) := \frac{b-a}{\log b - \log a};$$

$$I = I(a, b) := (b^b/a^a)^{1/(b-a)}/e; \quad A = A(a, b) := \frac{a+b}{2}; \quad S = S(a, b) := a^{\frac{a}{a+b}} b^{\frac{b}{a+b}},$$

are the harmonic, geometric, logarithmic, identric, arithmetic and Gini mean, respectively.

An easy task is to construct intermediate means related to two given means  $M$  and  $N$  with  $M \leq N$ . For instance, for an arbitrary mean  $P$ , we have that

$$M(a, b) \leq P(M(a, b), N(a, b)) \leq N(a, b).$$

The problem is more difficult if we have to decide whether the given mean is intermediate or not. For example, the relation

$$L(a, b) \leq S_s(a, b) \leq I(a, b),$$

holds for each positive  $a$  and  $b$  if and only if  $0 \leq s \leq 1$ , where the Stolarsky mean  $S_s$  is defined by (cf [4])

$$S_s(a, b) := \left( \frac{b^s - a^s}{s(b - a)} \right)^{1/(s-1)}.$$

Also,

$$G(a, b) \leq A_s(a, b) \leq A(a, b),$$

holds if and only if  $0 \leq s \leq 1$ , where the Hölder mean of order  $s$  is defined by

$$A_s(a, b) := \left( \frac{a^s + b^s}{2} \right)^{1/s}.$$

An inverse problem is to find best possible approximation of a given mean  $P$  by elements of an ordered class of means  $S$ . A good example for this topic is comparison between the logarithmic mean and the class  $A_s$  of Hölder means of order  $s$ . Namely, since  $A_0 = \lim_{s \rightarrow 0} A_s = G$  and  $A_1 = A$ , it follows from (1) that

$$A_0 \leq L \leq A_1.$$

Since  $A_s$  is monotone increasing in  $s$ , an improving of the above is given by Carlson [2]:

$$A_0 \leq L \leq A_{1/2}.$$

Finally, Lin showed in [3] that

$$A_0 \leq L \leq A_{1/3},$$

is the best possible approximation of the logarithmic mean by the means from the class  $A_s$ .

Numerous similar results have been obtained recently. For example, an approximation of Seiffert's mean by the class  $A_s$  is given in [6], [8].

In this article we shall give best possible approximations for a whole variety of elementary means (1) by the class  $\lambda_s$  defined below (see Thm 3.).

**1. 2.** Let  $f, g$  be twice continuously differentiable (strictly) convex functions on  $\mathbb{R}^+$ . By definition (cf [1], p. 5),

$$\bar{f}(a, b) := f(a) + f(b) - 2f\left(\frac{a+b}{2}\right) > 0, \quad a \neq b,$$

and

$$\bar{f}(a, b) = 0,$$

if and only if  $a = b$ .

It turns out that the expression

$$\Lambda_{f,g}(a, b) := \frac{\bar{f}(a, b)}{\bar{g}(a, b)} = \frac{f(a) + f(b) - 2f\left(\frac{a+b}{2}\right)}{g(a) + g(b) - 2g\left(\frac{a+b}{2}\right)},$$

represents a mean of two positive numbers  $a, b$ ; that is, the relation

$$\min\{a, b\} \leq \Lambda_{f,g}(a, b) \leq \max\{a, b\}, \quad (2)$$

holds for each  $a, b \in \mathbb{R}^+$ , if and only if the relation

$$f''(t) = tg''(t), \quad (3)$$

holds for each  $t \in \mathbb{R}^+$ .

Let  $f, g \in C^\infty(0, \infty)$  and denote by  $\Lambda$  the set  $\{(f, g)\}$  of convex functions satisfying the relation (3). There is a natural question how to improve the bounds in (2); in this sense we come upon the following intermediate mean problem:

**Open question** *Under what additional conditions on  $f, g \in \Lambda$ , the inequalities*

$$H(a, b) \leq \Lambda_{f,g}(a, b) \leq A(a, b),$$

*or, more tightly,*

$$L(a, b) \leq \Lambda_{f,g}(a, b) \leq I(a, b),$$

*hold for each  $a, b \in \mathbb{R}^+$ ?*

As an illustration, consider the function  $f_s(t)$  defined to be

$$f_s(t) = \begin{cases} (t^s - st + s - 1)/s(s - 1) & , s(s - 1) \neq 0; \\ t - \log t - 1 & , s = 0; \\ t \log t - t + 1 & , s = 1. \end{cases}$$

Since

$$f'_s(t) = \begin{cases} \frac{t^{s-1}-1}{s-1} & , s(s-1) \neq 0; \\ 1 - \frac{1}{t} & , s = 0; \\ \log t & , s = 1, \end{cases}$$

and

$$f''_s(t) = t^{s-2}, \quad s \in \mathbb{R}, \quad t > 0,$$

it follows that  $f_s(t)$  is a twice continuously differentiable convex function for  $s \in \mathbb{R}$ ,  $t \in \mathbb{R}^+$ .

Moreover, it is evident that  $(f_{s+1}, f_s) \in \Lambda$ .

We shall give in the sequel a complete answer to the above question concerning the means

$$\bar{f}_{s+1}(a, b)/\bar{f}_s(a, b) := \lambda_s(a, b)$$

defined by

$$\lambda_s(a, b) = \begin{cases} \frac{s-1}{s+1} \frac{a^{s+1}+b^{s+1}-2(\frac{a+b}{2})^{s+1}}{a^s+b^s-2(\frac{a+b}{2})^s}, & s \in \mathbb{R}/\{-1, 0, 1\}; \\ \frac{2 \log \frac{a+b}{2} - \log a - \log b}{\frac{1}{2a} + \frac{1}{2b} - \frac{2}{a+b}}, & s = -1; \\ \frac{a \log a + b \log b - (a+b) \log \frac{a+b}{2}}{2 \log \frac{a+b}{2} - \log a - \log b}, & s = 0; \\ \frac{(b-a)^2}{4(a \log a + b \log b - (a+b) \log \frac{a+b}{2})}, & s = 1. \end{cases}$$

Those means are obviously symmetric and homogeneous of order one.

As a consequence we obtain some new intermediate mean values; for instance, we show that the inequalities

$$H(a, b) \leq \lambda_{-1}(a, b) \leq G(a, b) \leq \lambda_0(a, b) \leq L(a, b) \leq \lambda_1(a, b) \leq I(a, b),$$

hold for arbitrary  $a, b \in \mathbb{R}^+$ .

Note that

$$\lambda_{-1} = \frac{2G^2 \log(A/G)}{A - H}; \quad \lambda_0 = A \frac{\log(S/A)}{\log(A/G)}; \quad \lambda_1 = \frac{1}{2} \frac{A - H}{\log(S/A)}.$$

## 2. Results

We prove firstly the following

**Theorem 1** *Let  $f, g \in C^2(I)$  with  $g'' > 0$ . The expression  $\Lambda_{f,g}(a, b)$  represents a mean of arbitrary numbers  $a, b \in I$  if and only if the relation*

$$f''(t) = tg''(t) \tag{3}$$

*holds for  $t \in I$ .*

**Remark 1** *In the same way, for arbitrary  $p, q > 0, p + q = 1$ , it can be deduced that the quotient*

$$\Lambda_{f,g}(p, q; a, b) := \frac{pf(a) + qf(b) - f(pa + qb)}{pg(a) + qg(b) - g(pa + qb)}$$

*represents a mean value of numbers  $a, b$  if and only if (3) holds.*

A generalization of the above assertion is the next

**Theorem 2** *Let  $f, g : I \rightarrow \mathbb{R}$  be twice continuously differentiable functions with  $g'' > 0$  on  $I$  and let  $p = \{p_i\}, i = 1, 2, \dots, \sum p_i = 1$  be an arbitrary positive weight sequence. Then the quotient of two Jensen functionals*

$$\Lambda_{f,g}(p, x) := \frac{\sum_1^n p_i f(x_i) - f(\sum_1^n p_i x_i)}{\sum_1^n p_i g(x_i) - g(\sum_1^n p_i x_i)}, \quad n \geq 2,$$

*represents a mean of an arbitrary set of real numbers  $x_1, x_2, \dots, x_n \in I$  if and only if the relation*

$$f''(t) = tg''(t)$$

*holds for each  $t \in I$ .*

**Remark 2** *It should be noted that the relation  $f''(t) = tg''(t)$  determines  $f$  in terms of  $g$  in an easy way. Precisely,*

$$f(t) = tg(t) - 2G(t) + ct + d,$$

*where  $G(t) := \int_1^t g(u)du$  and  $c$  and  $d$  are constants.*

Our results concerning the means  $\lambda_s(a, b)$ ,  $s \in \mathbb{R}$  are included in the following

**Theorem 3** *For the class of means  $\lambda_s(a, b)$  defined above, the following assertions hold for each  $a, b \in \mathbb{R}^+$ .*

1. The means  $\lambda_s(a, b)$  are monotone increasing in  $s$ ;
2.  $\lambda_s(a, b) \leq H(a, b)$  for each  $s \leq -4$ ;
3.  $H(a, b) \leq \lambda_s(a, b) \leq G(a, b)$  for  $-3 \leq s \leq -1$ ;
4.  $G(a, b) \leq \lambda_s(a, b) \leq L(a, b)$  for  $-1/2 \leq s \leq 0$ ;
5. there is a number  $s_0 \in (1/12, 1/11)$  such that  $L(a, b) \leq \lambda_s(a, b) \leq I(a, b)$  for  $s_0 \leq s \leq 1$ ;
6. there is a number  $s_1 \in (1.03, 1.04)$  such that  $I(a, b) \leq \lambda_s(a, b) \leq A(a, b)$  for  $s_1 \leq s \leq 2$ ;
7.  $A(a, b) \leq \lambda_s(a, b) \leq S(a, b)$  for each  $2 \leq s \leq 5$ ;
8. there is no finite  $s$  such that the inequality  $S(a, b) \leq \lambda_s(a, b)$  holds for each  $a, b \in \mathbb{R}^+$ .

The above estimations are best possible.

### 3. Proofs

**Proof of Theorem 1** We prove firstly the necessity of the condition (3).

Since  $\Lambda_{f,g}(a, b)$  is a mean value for arbitrary  $a, b \in I$ ;  $a \neq b$ , we have

$$\min\{a, b\} \leq \Lambda_{f,g}(a, b) \leq \max\{a, b\}.$$

Hence

$$\lim_{b \rightarrow a} \Lambda_{f,g}(a, b) = a. \quad (4)$$

From the other hand, due to l'Hospital's rule we obtain

$$\begin{aligned} \lim_{b \rightarrow a} \Lambda_{f,g}(a, b) &= \lim_{b \rightarrow a} \left( \frac{f'(b) - f'(\frac{a+b}{2})}{g'(b) - g'(\frac{a+b}{2})} \right) = \lim_{b \rightarrow a} \left( \frac{2f''(b) - f''(\frac{a+b}{2})}{2g''(b) - g''(\frac{a+b}{2})} \right) \\ &= \frac{f''(a)}{g''(a)}. \end{aligned} \quad (5)$$

Comparing (4) and (5) the desired result follows.

Suppose now that (3) holds and let  $a < b$ . Since  $g''(t) > 0$   $t \in [a, b]$  by the *Cauchy mean value theorem* there exists  $\xi \in (\frac{a+t}{2}, t)$  such that

$$\frac{f'(t) - f'(\frac{a+t}{2})}{g'(t) - g'(\frac{a+t}{2})} = \frac{f''(\xi)}{g''(\xi)} = \xi. \quad (6)$$

But,

$$a \leq \frac{a+t}{2} < \xi < t \leq b,$$

and, since  $g'$  is strictly increasing,  $g'(t) - g'(\frac{a+t}{2}) > 0$ ,  $t \in [a, b]$ .

Therefore, by (6) we get

$$a(g'(t) - g'(\frac{a+t}{2})) \leq f'(t) - f'(\frac{a+t}{2}) \leq b(g'(t) - g'(\frac{a+t}{2})). \quad (7)$$

Finally, integrating (7) over  $t \in [a, b]$  we obtain the assertion from Theorem 1.

**Proof of Theorem 2** We shall give a proof of this assertion by induction on  $n$ .

By Remark 1, it holds for  $n = 2$ .

Next, it is not difficult to check the identity

$$\begin{aligned} \sum_1^n p_i f(x_i) - f(\sum_1^n p_i x_i) &= (1 - p_n) \left( \sum_1^{n-1} p'_i f(x_i) - f(\sum_1^{n-1} p'_i x_i) \right) \\ &\quad + [(1 - p_n)f(T) + p_n f(x_n) - f((1 - p_n)T + p_n x_n)], \end{aligned}$$

where

$$T := \sum_1^{n-1} p'_i x_i; \quad p'_i := p_i / (1 - p_n), \quad i = 1, 2, \dots, n-1; \quad \sum_1^{n-1} p'_i = 1.$$

Therefore, by induction hypothesis and Remark 1, we get

$$\begin{aligned} \sum_1^n p_i f(x_i) - f(\sum_1^n p_i x_i) &\leq \max\{x_1, x_2, \dots, x_{n-1}\} (1 - p_n) \left( \sum_1^{n-1} p'_i g(x_i) - g(\sum_1^{n-1} p'_i x_i) \right) \\ &\quad + \max\{T, x_n\} [(1 - p_n)g(T) + p_n g(x_n) - g((1 - p_n)T + p_n x_n)] \\ &\leq \max\{x_1, x_2, \dots, x_n\} ((1 - p_n) \left( \sum_1^{n-1} p'_i g(x_i) - g(\sum_1^{n-1} p'_i x_i) \right) \\ &\quad + [(1 - p_n)g(T) + p_n g(x_n) - g((1 - p_n)T + p_n x_n)]) \\ &= \max\{x_1, x_2, \dots, x_n\} \left( \sum_1^n p_i g(x_i) - g(\sum_1^n p_i x_i) \right). \end{aligned}$$

The inequality

$$\min\{x_1, x_2, \dots, x_n\} \leq \Lambda_{f,g}(p, x),$$

can be proved analogously.

For the proof of necessity, put  $x_2 = x_3 = \dots = x_n$  and proceed as in Theorem 1.

**Remark** It is evident from (3) that if  $I \subseteq \mathbb{R}^+$  then  $f$  has to be also convex on  $I$ . Otherwise, it shouldn't be the case. For example, the conditions of Theorem 2 are satisfied with  $f(t) = t^3/3, g(t) = t^2, t \in \mathbb{R}$ . Hence, for an arbitrary sequence  $\{x_i\}_1^n$  of real numbers, we obtain

$$\min\{x_1, x_2, \dots, x_n\} \leq \frac{\sum_1^n p_i x_i^3 - (\sum_1^n p_i x_i)^3}{3(\sum_1^n p_i x_i^2 - (\sum_1^n p_i x_i)^2)} \leq \max\{x_1, x_2, \dots, x_n\}.$$

Because the above inequality does not depend on  $n$ , a probabilistic interpretation of the above result is contained in the following

**Theorem 4.** *For an arbitrary probability law  $F$  of random variable  $X$  with support on  $(-\infty, +\infty)$ , we have*

$$(EX)^3 + 3(\min X) \sigma_X^2 \leq EX^3 \leq (EX)^3 + 3(\max X) \sigma_X^2.$$

**Proof of Theorem 3, part 1** We shall prove a general assertion of this type. Namely, for an arbitrary positive sequence  $\mathbf{x} = \{x_i\}$  and an associated weight sequence  $\mathbf{p} = \{p_i\}$ ,  $i = 1, 2, \dots$ , denote

$$\chi_s(\mathbf{p}, \mathbf{x}) := \begin{cases} \frac{\sum p_i x_i^s - (\sum p_i x_i)^s}{s(s-1)}, & s \in \mathbb{R}/\{0, 1\}; \\ \log(\sum p_i x_i) - \sum p_i \log x_i, & s = 0; \\ \sum p_i x_i \log x_i - (\sum p_i x_i) \log(\sum p_i x_i), & s = 1. \end{cases}$$

For  $s \in \mathbb{R}, r > 0$  we have

$$\chi_s(\mathbf{p}, \mathbf{x}) \chi_{s+r+1}(\mathbf{p}, \mathbf{x}) \geq \chi_{s+1}(\mathbf{p}, \mathbf{x}) \chi_{s+r}(\mathbf{p}, \mathbf{x}), \quad (4)$$

which is equivalent to

**Theorem 3a** *The sequence  $\{\chi_{s+1}(\mathbf{p}, \mathbf{x})/\chi_s(\mathbf{p}, \mathbf{x})\}$  is monotone increasing in  $s$ ,  $s \in \mathbb{R}$ .*

This assertion follows applying the result from ([5], Theorem 2) which states that

**Lemma 1** *For  $-\infty < a < b < c < +\infty$ , the inequality*

$$(\chi_b(\mathbf{p}, \mathbf{x}))^{c-a} \leq (\chi_a(\mathbf{p}, \mathbf{x}))^{c-b} (\chi_c(\mathbf{p}, \mathbf{x}))^{b-a},$$

*holds for arbitrary sequences  $\mathbf{p}, \mathbf{x}$ .*

Putting there  $a = s, b = s + 1, c = s + r + 1$  and  $a = s, b = s + r, c = s + r + 1$ , we successively obtain

$$(\chi_{s+1}(\mathbf{p}, \mathbf{x}))^{r+1} \leq (\chi_s(\mathbf{p}, \mathbf{x}))^r \chi_{s+r+1}(\mathbf{p}, \mathbf{x}),$$

and

$$(\chi_{s+r}(\mathbf{p}, \mathbf{x}))^{r+1} \leq \chi_s(\mathbf{p}, \mathbf{x}) (\chi_{s+r+1}(\mathbf{p}, \mathbf{x}))^r.$$

Since  $r > 0$ , multiplying those inequalities we get the relation (4) i. e. the proof of Theorem 3a.

The part 1. of Theorem 3 follows for  $p_1 = p_2 = 1/2$ .

A general way to prove the rest of Theorem 3 is to use an easy-checkable identity

$$\frac{\lambda_s(a, b)}{A(a, b)} = \lambda_s(1 + t, 1 - t),$$

with  $t := \frac{b-a}{b+a}$ .

Since  $0 < a < b$ , we get  $0 < t < 1$ . Also,

$$\frac{H(a, b)}{A(a, b)} = 1 - t^2; \quad \frac{G(a, b)}{A(a, b)} = \sqrt{1 - t^2}; \quad \frac{L(a, b)}{A(a, b)} = \frac{2t}{\log(1+t) - \log(1-t)}; \quad (5)$$

$$\frac{I(a, b)}{A(a, b)} = \exp\left(\frac{(1+t)\log(1+t) - (1-t)\log(1-t)}{2t} - 1\right); \quad \frac{S(a, b)}{A(a, b)} = \exp\left(\frac{1}{2}((1+t)\log(1+t) + (1-t)\log(1-t))\right).$$

Therefore, we have to compare some one-variable inequalities and to check their validness for each  $t \in (0, 1)$ .

For example, we shall prove that the inequality

$$\lambda_s(a, b) \leq L(a, b)$$

holds for each positive  $a, b$  if and only if  $s \leq 0$ .

Since  $\lambda_s(a, b)$  is monotone increasing in  $s$ , it is enough to prove that

$$\frac{\lambda_0(a, b)}{L(a, b)} \leq 1.$$

By the above formulae, this is equivalent to the assertion that the inequality

$$\phi(t) \leq 0 \quad (6)$$

holds for each  $t \in (0, 1)$ , with

$$\phi(t) := \frac{\log(1+t) - \log(1-t)}{2t}((1+t)\log(1+t) + (1-t)\log(1-t)) + \log(1+t) + \log(1-t).$$

We shall prove that the power series expansion of  $\phi(t)$  have non-positive coefficients. Thus the relation (6) will be proved.

Since

$$\frac{\log(1+t) - \log(1-t)}{2t} = \sum_0^{\infty} \frac{t^{2k}}{2k+1}; \quad \log(1+t) + \log(1-t) = -t^2 \sum_0^{\infty} \frac{t^{2k}}{k+1};$$

$$(1+t)\log(1+t) + (1-t)\log(1-t) = t^2 \sum_0^{\infty} \frac{t^{2k}}{(k+1)(2k+1)},$$

we get

$$\phi(t)/t^2 = \sum_{n=0}^{\infty} \left( -\frac{1}{n+1} + \sum_{k=0}^n \frac{1}{(2n-2k+1)(k+1)(2k+1)} \right) t^{2n} = \sum_0^{\infty} c_n t^{2n}.$$

Hence,

$$c_0 = c_1 = 0; \quad c_2 = -1/90,$$



and, after some calculation, we get

$$c_n = \frac{2}{(n+1)(2n+3)} \left( (n+2) \sum_1^n \frac{1}{2k+1} - (n+1) \sum_1^n \frac{1}{2k} \right), \quad n > 1.$$

Now, one can easily prove (by induction, for example) that

$$d_n := (n+2) \sum_1^n \frac{1}{2k+1} - (n+1) \sum_1^n \frac{1}{2k},$$

is a negative real number for  $n \geq 2$ . Therefore  $c_n \leq 0$ , and the proof of the first part is done.

For  $0 < s < 1$  we have

$$\frac{\lambda_s(a, b)}{L(a, b)} - 1 = \frac{(1-s)((1+t)^{s+1} + (1-t)^{s+1} - 2) \log \frac{1+t}{1-t}}{2t(1+s)(2 - (1+t)^s - (1-t)^s)} - 1 = \frac{1}{6}st^2 + O(t^4) \quad (t \rightarrow 0).$$

Therefore,  $\lambda_s(a, b) > L(a, b)$  for  $s > 0$  and sufficiently small  $t := (b-a)/(b+a)$ .

Similarly, we shall prove that the inequality

$$\lambda_s(a, b) \leq I(a, b),$$

holds for each  $a, b; 0 < a < b$  if and only if  $s \leq 1$ .

As before, it is enough to consider the expression

$$\frac{I(a, b)}{\lambda_1(a, b)} = e^{\mu(t)} \nu(t) := \psi(t),$$

with

$$\mu(t) = \frac{(1+t) \log(1+t) - (1-t) \log(1-t)}{2t} - 1; \quad \nu(t) = \frac{(1+t) \log(1+t) + (1-t) \log(1-t)}{t^2}.$$

It is not difficult to check the identity

$$\psi'(t) = -e^{\mu(t)} \phi(t)/t^3.$$

Hence by (6), we get  $\psi'(t) > 0$  i. e.  $\psi(t)$  is monotone increasing for  $t \in (0, 1)$ .

Therefore

$$\frac{I(a, b)}{\lambda_1(a, b)} \geq \lim_{t \rightarrow 0^+} \psi(t) = 1.$$

By monotonicity it follows that  $\lambda_s(a, b) \leq I(a, b)$  for  $s \leq 1$ .

For  $s > 1$ ,  $\frac{b-a}{b+a} = t$ , we have

$$\lambda_s(a, b) - I(a, b) = \left( \frac{1}{6}(s-1)t^2 + O(t^4) \right) A(a, b) \quad (t \rightarrow 0^+).$$

Hence,  $\lambda_s(a, b) > I(a, b)$  for  $s > 1$  and  $t$  sufficiently small.

From the other hand,

$$\lim_{t \rightarrow 1^-} \left[ \frac{\lambda_s(a, b)}{I(a, b)} - 1 \right] = \frac{e(s-1)(2^{s+1} - 2)}{2(s+1)(2^s - 2)} - 1 := \tau(s).$$

Examining the function  $\tau(s)$ , we find out that it has the only real zero at  $s_0 \approx 1.0376$  and is negative for  $s \in (1, s_0)$ .

**Remark 2** Since  $\psi(t)$  is monotone increasing, we also get

$$\frac{I(a, b)}{\lambda_1(a, b)} \leq \lim_{t \rightarrow 1^-} \psi(t) = \frac{4 \log 2}{e}.$$

Hence

$$1 \leq \frac{I(a, b)}{\lambda_1(a, b)} \leq \frac{4 \log 2}{e}.$$

A calculation gives  $\frac{4 \log 2}{e} \approx 1.0200$ .

Note also that

$$\lambda_2(a, b) \equiv A(a, b).$$

Therefore, applying the assertion from the part 1., we get

$$\lambda_s(a, b) \leq A(a, b), \quad s \leq 2; \quad \lambda_s(a, b) \geq A(a, b), \quad s \geq 2.$$

Finally, we give a detailed proof of the part 7.

We have to prove that  $\lambda_s(a, b) \leq S(a, b)$  for  $s \leq 5$ . Since  $\lambda_s(a, b)$  is monotone increasing in  $s$ , it is sufficient to prove that the inequality

$$\lambda_5(a, b) \leq S(a, b)$$

holds for each  $a, b \in \mathbb{R}^+$ .

Therefore, by the transformation given above, we get

$$\begin{aligned} \log \frac{\lambda_5}{A} &= \log \left[ \frac{2(1+t)^6 + (1-t)^6 - 2}{3(1+t)^5 + (1-t)^5 - 2} \right] = \log \left[ \frac{2}{15} \frac{15 + 15t^2 + t^4}{2 + t^2} \right] \\ &\leq \log \left[ \frac{1 + t^2 + t^4/4}{1 + t^2/2} \right] = \log(1 + t^2/2) = t^2/2 - t^4/8 + t^6/24 - \dots \\ &\leq t^2/2 + t^4/12 + t^6/30 + \dots = \frac{1}{2}((1+t) \log(1+t) + (1-t) \log(1-t)) = \log S/A, \end{aligned}$$

and the proof is done.

Further, we have to show that  $\lambda_s(a, b) > S(a, b)$  for some positive  $a, b$  whenever  $s > 5$ .

Indeed, since

$$(1+t)^s + (1-t)^s - 2 = \binom{s}{2}t^2 + \binom{s}{4}t^4 + O(t^6),$$

for  $s > 5$  and sufficiently small  $t$ , we get

$$\begin{aligned} \frac{\lambda_s}{A} &= \frac{s-1}{s+1} \frac{\binom{s+1}{2}t^2 + \binom{s+1}{4}t^4 + O(t^6)}{\binom{s}{2}t^2 + \binom{s}{4}t^4 + O(t^6)} \\ &= \frac{1+(s-1)(s-2)t^2/12 + O(t^4)}{1+(s-2)(s-3)t^2/12 + O(t^4)} = 1 + \left(\frac{s}{6} - \frac{1}{3}\right)t^2 + O(t^4). \end{aligned}$$

Similarly,

$$\frac{S}{A} = \exp\left(\frac{1}{2}((1+t)\log(1+t) + (1-t)\log(1-t))\right) = \exp(t^2/2 + O(t^4)) = 1 + t^2/2 + O(t^4).$$

Hence,

$$\frac{1}{A}(\lambda_s - S) = \frac{1}{6}(s-5)t^2 + O(t^4),$$

and this expression is positive for  $s > 5$  and  $t$  sufficiently small, i.e.  $a$  sufficiently close to  $b$ .

As for the part 8., applying the above transformation we obtain

$$\frac{\lambda_s(a, b)}{S(a, b)} = \frac{s-1}{s+1} \frac{(1+t)^{s+1} + (1-t)^{s+1} - 2}{(1+t)^s + (1-t)^s - 2} \exp\left(-\frac{1}{2}((1+t)\log(1+t) + (1-t)\log(1-t))\right),$$

where  $0 < a < b$ ,  $t = \frac{b-a}{b+a}$ .

Since for  $s > 5$ ,

$$\lim_{t \rightarrow 1^-} \frac{\lambda_s}{S} = \frac{s-1}{s+1} \frac{2^s - 1}{2^s - 2},$$

and the last expression is less than one, it follows that the inequality  $S(a, b) < \lambda_s(a, b)$  cannot hold whenever  $\frac{b}{a}$  is sufficiently large.

The rest of the proof is straightforward.

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